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On the Lie symmetry algebra of a general ordinary differential equation

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Abstract. We give a description of the Lie symmetry algebra of a general ordinary differential equation which clarifies certain issues in its practical use. In particular we present examples of symmetry algebras for well known equations which illustrate features of this algebra.

1. Introduction

In a recent publication [3] a complete description of the Lie symmetry algebra of a linear ordinary differential equation (ODE) with coefficients in a specified differential base field was given and this paper will generalize the result to an equation defined by an element of a differential ring in a single indeterminate over the base field and hence to nonlinear ODEs. A somewhat different description of the linear case is given in [17]. The generalization is straightforward in that the algebra is just the ring of derivations of a function space in certain invariants as, it will be shown, it is in the linear case.

The Lie symmetries of a differential equation are of immense practical help in understanding the structure of such an equation and in obtaining families of particular integrals. What we are aiming at is an abstract framework which might be compared with the differential Galois theory of a linear equation. Indeed, in the case of linear equations this comparison ought to be a correspondence. We do not therefore advance a practical tool in obtaining the symmetry algebra, which is in any case tantamount to being able to solve the equation completely, but to construct an object whose structure is transparent. By choosing certain classes of object we pin down classes of differential equation whose structure then becomes equally transparent.

In this paper we give the definition of the (non-characteristic) Lie symmetry algebra and present the main theorem, which encompasses the earlier result, before giving what we hope are interesting examples of its application. These are to do with subalgebras whose coefficients belong to subfields of the full field of invariants.

It must be noted that the non-characteristic Lie algebra is not exactly the same as the Lie symmetry algebra as defined in, say, [13] and is so-called because certain characteristic symmetries have been factored out of the latter. Further, the invariants we use are invariants of the characteristic symmetry. In addition, what we call the Lie symmetry algebra here is more broadly defined than in [13] where only *point* symmetries are intended. Our use is in accord with others [7, 16]. In defence of the broader definition we point out that when written in first-order system form (so that each derivative is effectively a new dependent variable) an ODE has ‘point’ symmetries which transform derivatives in ways which are not necessarily

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prologations of point transformations of the original dependent and independent variables. Such symmetries are often called generalized, dynamical or Lie–Bäcklund symmetries. The value of this point of view has been amply demonstrated in [9], where examples are given of families of second-order ODEs possessing no point symmetries but integrable by quadrature by virtue of dynamical symmetries. In [8] the authors make the point that, for second-order differential equations, integrability by quadrature is equivalent to the presence of a pair of dynamical symmetries.

2. Definitions and basic results

In this section we present the foundations of a coordinate-free treatment of generalized, continuous symmetries for differential systems of finite rank. The classical notion of Lie point symmetry is extended to give a larger algebra of vector fields (derivations) preserving the differential system (ideal) under Lie transport. We present results on reduction of order proved in [3] and use them to give a new result on solvability (theorem 9).

Let k be a differential field of characteristic zero with derivation ∂_x and field of constants k_0. $k[y]$ is the ring of polynomials over k in the indeterminates $y_0, y_1, \ldots, y_N$ with commuting derivations $\partial_{y_0}, \partial_{y_1}, \ldots, \partial_{y_N}$. $k[y]$ is an integral domain with fraction field denoted $k(y)$ to which $\partial_x, \partial_{y_0}, \ldots, \partial_{y_N}$ extend in the obvious way. For the $k(y)$ valued k_0-linear derivations on $k(y)$ we will write $\mathrm{Der}(k(y))$. These are $k(y)$ linear combinations of the above derivations.

We shall also want to adjoin to a general differential field, J say, new functions $j_1, j_2, \ldots$ defined by differential equations whose defining function belongs to K, a differential field extension of $k(y)$. In this way we get differential fields $J\langle j_1\rangle$, $J\langle j_1, j_2\rangle$ etc. The Picard–Vessiot extensions [12] are examples of this.

By $\bigwedge(k(y))$ we denote the $k(y)$ valued one-forms dual to $\mathrm{Der}(k(y))$. A basis is given by the set $\{\mathrm{d}x, \mathrm{d}y_1, \mathrm{d}y_2, \ldots, \mathrm{d}y_N\}$ where $\mathrm{d}x\lfloor\partial_x = 1$, $\mathrm{d}x\lfloor\partial_{y_i} = 0$ for all i and $\mathrm{d}y_i\lfloor\partial_{y_j} = \delta_{ij}$. (We use $\lfloor$ for the pairing between a vector space and its dual.) The exterior differential algebra will be denoted $\bigwedge(k(y)) = \oplus_0^N \bigwedge^p(k(y))$ with $\mathrm{d}a = \partial_x a\,\mathrm{d}x + \partial_{y_0} a\,\mathrm{d}y_0 + \partial_{y_1} a\,\mathrm{d}y_1 + \cdots$ for $a \in k(y)$.

An *ideal* of $\bigwedge(k(y))$ will be an algebra ideal which is also closed under the exterior derivative d. So if ω and η are elements of the algebra and the ideal respectively, both $\omega\wedge\eta$ and $\mathrm{d}\eta$ belong to the ideal. We use the notation $A \triangleleft B$ when A is an ideal of B. We will assume, because of the applications we have in mind, that our ideals are finitely generated by one-forms. Although the definition of symmetry algebra given below generalizes to partial differential equations, this restriction excludes most of them.

Let K be a differential field extension of $k(y)$. We consider forms, $\bigwedge(K)$, over K and the k_0-linear derivations, $\mathrm{Der}(K)$ of K. We have the following.

Definition 1. The K-Lie symmetry algebra, $\mathcal{L}_\Theta(K)$, of an ideal $\Theta \subset \bigwedge(K)$ is the k_0-algebra of vector fields $X \in \mathrm{Der}(K)$ such that $L_X\Theta \subseteq \Theta$.

The definition uses the Lie derivative with respect to X, namely, $L_X\theta = \mathrm{d}(X\lfloor\theta) + X\lfloor\mathrm{d}\theta$ for any form θ. The ideal is preserved by Lie transport along symmetries and hence integral manifolds are transported into integral manifolds. This is the natural dynamical or geometric generalization of Lie point symmetry.

Definition 2. The K-characteristic algebra, $\mathcal{X}_\Theta(K)$ of Θ is the k_0-algebra of vector fields $X \in \mathrm{Der}(K)$ such that $X\lfloor\Theta \subset \Theta$.

It is easily verified that, as k_0-algebras, $\mathcal{X}_\Theta \triangleleft \mathcal{L}_\Theta$.

Definition 3. The K-non-characteristic symmetry algebra, $\mathcal{S}_\Theta(K)$, of Θ is the k_0-algebra $\mathcal{L}_\Theta(K)/\mathcal{X}_\Theta(K)$.

In general this algebra will be infinite dimensional over k_0 but we will see that it is finite dimensional *as a vector space*, over a larger field. In factoring out the characteristic symmetries we are removing 'trivial' symmetries which simply transport each individual integral manifold into itself.

Definition 4. For K any field extension of $k(y)$, the K-invariants of Θ, $\mathcal{I}_\Theta(K)$ are those elements $a \in K$ such that $X \lfloor \mathrm{d}a = 0$ for all $X \in \mathcal{X}_\Theta(K)$.

From now on we will drop the argument from $\mathcal{S}_\Theta$ etc except where we wish to emphasize the particular field in question.

In order to discuss symmetry reduction of Θ, and hence the reduction in order of the differential system, we need to understand the relations between the full symmetry of Θ and the symmetries of ideals contained within Θ. The situation here is not as neat as one might wish. Nevertheless the following results lead up to the generalization to $\mathcal{S}_\Theta(K)$ of a well known result, theorem 9, a sufficient condition for 'integrability by quadrature' up a tower of differential field inclusions. In what follows K is an extension of $k(y)$.

Lemma 5. Assume that $\mathcal{S}_\Theta$ is finite-dimensional as a K-vector space. If $\mathcal{N}$ is a subalgebra of $\mathcal{S}_\Theta$, also of finite dimension as a K-vector space, then there exists a K-algebra ideal $\Phi \lhd \Theta$ with non-characteristic symmetry algebra $\mathcal{S}_\Phi$ containing a subalgebra isomorphic to $\mathcal{I}(\mathcal{N})/\mathcal{N}$, $\mathcal{I}(\mathcal{N})$ being the idealizer of $\mathcal{N}$ in $\mathrm{Der}(K)$.

The ideal Φ in this theorem is generated over K by one-forms in Θ killed by elements of $\mathcal{N}$. In other words, it is the ideal of Θ for which $\mathcal{N}$ is characteristic. Recall that the idealizer of $\mathcal{N}$ is the largest subalgebra of $\mathrm{Der}(K)$ containing $\mathcal{N}$ as an ideal.

Definition 6. An ideal is *simple* if it is generated by one-forms and its *rank* is r if it is generated by r such forms linearly independent over K.

Lemma 7. An ideal, Θ, of rank one over K with a non-characteristic symmetry is generated by a closed one-form.

Lemma 8. If the single vector field $s \in \mathcal{S}_\Theta$ generates a subalgebra $\mathcal{N}$ then in the context of lemma 5, Θ/Φ has a closed generator.

These three lemmas are proved in [3].

Theorem 9. Let Θ have rank n over K and $\mathcal{S}_\Theta$ dimension n as a vector space over K. Suppose further that $\mathcal{S}_\Theta$ is solvable with derived series of length n. Then Θ has a decomposition of length n.

Proof. Let $\mathcal{S}_\Theta$ decompose thus:

$$\mathcal{S}_\Theta = \mathcal{S}_n \rhd \mathcal{S}_{n-1} \rhd \cdots \rhd \mathcal{S}_1 \rhd \mathcal{S}_0 = \{0\}. \tag{1}$$

For $s_1 \in \mathcal{S}_1$ let Φ_1 be generated by one-forms $\theta_2, \theta_3, \ldots, \theta_n$ such that $s_1 \lfloor \theta_i = 0$ for $i = 2, \ldots, n$. Choose $\theta_1 \notin \Phi_1$. Then $\theta_1' = (s_1 \lfloor \theta_1)^{-1}\theta_1$ is closed modulo Φ_1. Further, because the idealizer of $\mathcal{S}_1$ contains $\mathcal{S}_2$ we have that $\mathcal{S}_{\Phi_1} \supseteq \mathcal{S}_2/\mathcal{S}_1 \neq \{0\}$. Now choose $s_2 \in \mathcal{S}_2 \backslash \mathcal{S}_1$ and let Φ_2 be generated by those one-forms, say, $\theta_3, \ldots, \theta_n$ in Φ_1 with $s_2 \lfloor \theta_i = 0$ for $i = 3, \ldots, n$. Then $\theta_2' = (s_2 \lfloor \theta_2)^{-1}\theta_2$ is closed modulo Φ_2. This process continues to generate a chain of the form

$$\{0\} = \Phi_n \lhd \Phi_{n-1} \lhd \cdots \lhd \Phi_1 \lhd \Phi_0 = \Theta. \tag{2}$$

In this case we can solve the differential equation by successive quadratures up a tower of fields in the following manner. The theorem above has reduced the closure relations in the basis $\{\theta_1, \ldots, \theta_n\}$ to upper-triangular form. Consequently θ_n is closed and we may write $\theta_n = \mathrm{d} f_n$, at least locally, for some f_n in a large enough extension of $k(y)$. Restriction to the submanifold f_n constant then implies that the restricted form θ_{n-1} is closed, so $\theta_{n-1} = \mathrm{d} f_{n-1}$ where f_{n-1} is defined on the submanifold f_n constant and lives in a further extension of $k(y)$. Restricting to f_n and f_{n-1} both constant we repeat the procedure and iterating finally obtaining a set of n relations on the variables $x, y, y_1, \ldots, y_{n-1}$ and hence a (local) solution curve depending on the choice of constants. □

3. The main result

Let U be an open subset of $\mathbb{C}^{n+1}$ and let $\mathcal{A}(U)$ be the differential ring of functions analytic on U. As a commutative domain $\mathcal{A}$ has a field of fractions to which the derivations on $\mathcal{A}$ extend by the quotient rule. We call this the *analytic field* $\mathcal{F}(U)$. Suppose that Θ is an $\mathcal{F}(U)$-algebra ideal of rank n everywhere on U. By the Frobenius theorem there exist n functions, $I_1, I_2, \ldots, I_n \in \mathcal{F}(U')$, for a certain open subset $U' \subseteq U$, such that $\mathcal{F}(U')\langle\theta_1, \ldots, \theta_n\rangle = \mathcal{F}(U')\langle \mathrm{d}I_1, \mathrm{d}I_2, \ldots, \mathrm{d}I_n\rangle$. The I_i are analytic on U' and the $\mathrm{d}I_i$ are linearly independent over $\mathcal{F}(U')$. Consequently we can solve for $y_0, \ldots, y_{n-1}$ in terms of $x, I_1, \ldots, I_n$,

$$\begin{aligned} y_0 &= \Phi_0(x, I_1, \ldots, I_n) \\ &\vdots \\ y_{n-1} &= \Phi_{n-1}(x, I_1, \ldots, I_n) \end{aligned}$$

where the Φ_i are analytic on some open subset W of $\mathbb{C}^{n+1}$. This gives us an invertible, analytic map $\Phi : W \to U'$. Φ also induces a map from $\mathcal{F}(U')$ to $\mathcal{F}(W)$. Correspondingly the invertible tangent map, $\mathrm{d}\Phi$ maps derivations on $\mathcal{F}(W)\langle \mathrm{d}I_1, \ldots, \mathrm{d}I_n\rangle$ to derivations on $\mathcal{F}(U')\langle\theta_1, \ldots, \theta_n\rangle$. The characteristic derivation, X, on $\mathcal{F}(U')\langle\theta_1, \ldots, \theta_n\rangle$ satisfies $X\rfloor\theta_i = 0$ for $i = 1, \ldots, n$. and $d\Phi^{-1}(X)$ must satisfy $\mathrm{d}\Phi^{-1}(X)\rfloor \mathrm{d}I_i = 0$ for $i = 1, \ldots, n$. The latter has to be a multiple of ∂_x. Forming the factor algebra $\mathcal{L}_\Theta/\mathcal{X}_\Theta$ in each case we see that $\mathcal{S}_\Theta$ is mapped into the derivations of quotients of functions in $I_1, \ldots, I_n$ only, on W.

Thus we state the following theorem.

Theorem 10. If Θ is an $\mathcal{F}(U)$ ideal of maximal rank everywhere on U then there is an open $U' \subseteq U$ where $\mathcal{S}_\Theta(\mathcal{F}(U'))$ is the derivations of an analytic field of invariants.

4. Some illustrations

We illustrate the above with a range of examples.

4.1. Linear first-order ODE

It is often remarked that the symmetries in this instance form an infinite-dimensional Lie algebra which we describe below. In fact in the general linear case (see below) the non-characteristic Lie algebra is infinite-dimensional over its field of constants but finitely generated over the invariants by point symmetries.

Consider the first-order, linear equation, $y_1 - ay_0 = 0$, with $a \in k$, and let $y_0 = \alpha$ be any non-trivial solution. $k\langle\alpha\rangle = k_1$ is the smallest extension containing α. By a formal calculation it is easy to check that the general form of an element of $\mathcal{S}_\Theta$ is $y_0 F(y_0/\alpha)\partial_{y_0}$. Either $k = k_1$ or $k \subset k_1$ and k_1 is transcendental over k by lemma 3.9 of [12]. In the latter case $\mathcal{S}_\Theta(k(y_0))$ is one-dimensional over k_0 and generated by $y_0\partial_{y_0}$.

If we take k_1, the Liouville extension of k defined by the first-order, linear equation, we have an infinite set of symmetries. They can be expanded as Laurent series in y_0 whose terms are k_0-linear combinations of $s_i = y^{i+1}\alpha^{-i}\partial_{y_0}$ for $i \in \mathbb{Z}$. These s_i show that $\mathcal{S}_\Theta$ is, in this case, the k_0-algebra with Lie product,

$$[s_i, s_j] = (j - i)s_{i+j}. \tag{3}$$

4.2. General linear ODE

Let $z_1, z_2, \ldots, z_n$ be a k_0-linearly independent set of solutions to the given linear ODE of order n. We take k_1 to be the Picard–Vessiot extension of k defined by these functions. Then the n elements of $k_1(y)$ defined by the determinants

$$I_i = \begin{vmatrix} z_1 & \ldots & z_{i-1} & y_0 & z_{i+1} & \ldots & z_n \\ z_1^{(1)} & \ldots & z_{i-1}^{(1)} & y_1 & z_{i+1}^{(1)} & \ldots & z_n^{(1)} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ z_1^{(n-1)} & \ldots & z_{i-1}^{(n-1)} & y_{n-1} & z_{i+1}^{(n-1)} & \ldots & z_n^{(n-1)} \end{vmatrix} \tag{4}$$

are invariants of $\mathcal{X}_\Theta$, that is, $\mathrm{d}I_i \in \Theta$ and $\mathcal{S}_\Theta(k_1(y))$ is the k_0-algebra of derivations of the field $\mathcal{I}_\Theta = k_0(I_1, I_2, \ldots, I_n)$, [3]. This is a particular case of theorem 10. It is straightforward to verify that the $k_1(y)$-derivations of $\mathcal{I}_\Theta$ are $\mathcal{I}_\Theta$ generated by the n vector fields $X_i = z_i\partial_{y_0} + z_i^{(1)}\partial_{y_1} + \cdots + z_i^{(n-1)}\partial_{y_{n-1}}$.

An interesting observation is the relation between symmetry and factorization of linear operators. Suppose we are able to factorize an nth-order, linear operator $L = (\partial - a_1)(\partial - a_2)\ldots(\partial - a_n)$ over a differential field K. So $a_i \in K$ for $i = 1, \ldots, n$. The corresponding ideal Θ is generated by one-forms $\theta_i = \mathrm{d}u_{i-1} - (u_i + a_{n+1-i}u_{i-1})\,\mathrm{d}x$ for $i = 1, \ldots, n-1$ and $\theta_n = \mathrm{d}u_{n-1} - a_1 u_{n-1}\,\mathrm{d}x$. The closure relations of these basis elements have triangular form:

$$\mathrm{d}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} = \begin{bmatrix} a_n\,\mathrm{d}x & \mathrm{d}x & 0 & & \\ 0 & a_{n-1}\,\mathrm{d}x & \mathrm{d}x & & \\ & & \ddots & & \\ & & & a_2\,\mathrm{d}x & \mathrm{d}x \\ & & & 0 & a_1\,\mathrm{d}x \end{bmatrix} \wedge \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}. \tag{5}$$

By extending K if necessary we can reduce this to strictly upper-triangular form using, as integrating factors, the solutions to the equations $u' + a_p u = 0$. Consequently we have a chain of differential ideals, each describing a linear ODE:

$$\Theta_1 \rhd \Theta_2 \rhd \cdots \rhd \Theta_{n-1} \rhd \Theta_n = \Theta.$$

Θ_i arises as a symmetry reduction of Θ_{i+1}.

Suppose, in general, that D is a linear differential operator over a field k which factorizes, $D = PQ$, into linear differential operators over a field k'. Then we have a corresponding sequence of ideals:

$$\Theta_P \to \Theta_D \to \Theta_D/\Theta_P \sim \Theta_Q.$$

The factor ideal is equivalent to Θ_Q in the sense that on regular solution manifolds of Θ_P it restricts to a linear equation with an inhomogeneous term determined by the solution manifold chosen. If we take a symmetry reduction of the nth order operator D by one of the X_i defined above then P is an operator of order $n-1$ and Θ_Q represents a first-order, linear equation of the form $y_1 - \alpha y_0 = f$ for $f \in \ker P$.

4.3. Nonlinear first-order ODE

In this and subsequent sections where examples are drawn from equations of order of no more than two it is convenient to use y instead of y_0 and p instead of y_1. We use primes for x-derivatives.

We will illustrate this case using the general Ricatti equation because, for an explicit description of the symmetry algebra, one needs some sort of description of the solution space. The fact that the Ricatti equation is 'linearizable' to a second-order, linear equation, for which we know the symmetry algebra, is not *a priori* useful. We shall study the relationship between the two symmetry algebras in a moment.

Given any three projectively independent solutions u_1, u_2 and u_3 to the Riccati equation,

$$y' = y^2 + a(x) \tag{6}$$

the invariants in $k\langle u_1, u_2, u_3\rangle(y)$ are all rational functions of the cross ratio

$$z = \frac{(y-u_1)(u_3-u_2)}{(u_3-u_1)(y-u_2)}.$$

It is easy to check that any such function is killed by the characteristic field $\partial_x + (y^2+a)\partial_y$ and, by solving for y in terms of z that any function $F(x,z)$ is invariant only if independent of x. So $\mathfrak{I}(\Theta) = k_0(z)$. The general element of the non-characteristic symmetry algebra is then $h(z)(\frac{\partial z}{\partial y})^{-1}\partial_y$, that is, any derivation of the field of invariants. (For the appropriate invariant, z, this formula applies to any first-order equation.)

The existence of a symmetry implies the solution of the Ricatti equation by quadrature. If we use the symmetry $v = \frac{(y-u_1)(y-u_2)}{u_1-u_2}\partial_y$ with $\theta = \mathrm{d}y - (y^2+a)\,\mathrm{d}x$ it follows that the one-form

$$\frac{\theta}{v\rfloor\theta} = \frac{\mathrm{d}y - u_1'\,\mathrm{d}x}{y-u_1} - \frac{\mathrm{d}y - u_2'\,\mathrm{d}x}{y-u_2} + (u_2-u_1)\,\mathrm{d}x$$

is exact. This does not contradict the classical result on the impossibility of quadrature for the Ricatti equation because the classical result assumes only algebraic operations over $k(x,y)$. Indeed, the above is equivalent to the result that the Ricatti equation is integrable by one quadrature, the rightmost term above, when a pair of particular solutions is known.

To understand the relationship with the symmetry algebra of the corresponding second-order, linear equation we must prolong $\Theta = k(y)\langle \mathrm{d}y - (y^2+a)\,\mathrm{d}x\rangle$ to $\tilde{\Theta} = \tilde{k}(y,\phi)\langle \mathrm{d}y - (y^2+a)\,\mathrm{d}x, \mathrm{d}\phi + y\phi\,\mathrm{d}x\rangle$ which describes the second-order equation

$$\phi'' + a\phi = 0.$$

We have to relate the derivations of $k_0(z)$ to those of $k_0(I_1, I_2)$ where I_1 and I_2 are as described in (4). In terms of y and ϕ and a pair of linearly independent solutions ϕ_1 and ϕ_2 to the linear, second-order equation, the invariants I_1 and I_2 are given by

$$I_1 = -\phi(\phi_2' + y\phi_2)$$

and

$$I_2 = \phi(\phi_1' + y\phi_1).$$

The corresponding derivations are

$$X_1 = \phi_1 \partial_\phi - \frac{\phi_1' + y\phi_1}{\phi} \partial_y$$

and

$$X_2 = \phi_2 \partial_\phi - \frac{\phi_2' + y\phi_2}{\phi} \partial_y.$$

Now define u_1 and u_2, solutions to the Ricatti equation, by $\phi_1' = -u_1\phi_1$ and $\phi_2' = -u_2\phi_2$. A third solution to the Ricatti equation is built from these:

$$y_3 = -\frac{\phi_1' + c\phi_2'}{\phi_1 + c\phi_2}$$

c being an arbitrary (even infinite) constant. Rewriting the invariant z in terms of ϕ_1 and ϕ_2 gives

$$z = -\frac{1}{c} \frac{\phi_1' + y\phi_1}{\phi_2' + y\phi_2}.$$

Up to a constant this is the ratio of I_1 to I_2 and so $\mathcal{I}_{\Theta_1} = k_0(I_1/I_2) \subset k_0(I_1, I_2) = \mathcal{I}_\Theta$. The relationship between the non-characteristic symmetry algebras thus reduces to the study of the algebras of derivations of a field and a subfield. Generally speaking a derivation of the field is not necessarily a derivation of the subfield, or *vice versa*. In this case the subfield is fixed by the special derivation $I_1X_1 + I_2X_2$ which is to be expected since this is the symmetry of any linear, second-order equation and the one by which $\tilde{\Theta}$ is reduced to Θ. $\mathcal{I}_\Theta$ is thus playing a role anlagous to an intermediate fixed field in the Galois theory.

4.4. *Solvable structures*

A solvable structure, introduced in [6] and written in the current form in [11], is a chain of differential ideals

$$0 = \Theta_0 \triangleright \Theta_1 \triangleright \Theta_2 \triangleright \cdots \triangleright \Theta_{n-1} \triangleright \Theta_n = \Theta$$

with a special choice of one-forms $\theta_i \in \Theta_i \backslash \Theta_{i-1}$ such that $\mathrm{d}\theta_i \in \Theta_{i-1}$. The structure of the factorized, linear equation is a simple case. With each ideal Θ_i there is associated a symmetry algebra L_{Θ_i} and a non-characteristic symmetry algebra S_{Θ_i}. As we have seen there is no generic relationship between the algebras of the ith and $i-1$th ideals. Although the inclusion $\Theta_{i-1} \hookrightarrow \Theta_i$ can be regarded as coming from a symmetry reduction for some symmetry in S_{Θ_i}, this need not be a symmetry of $S_{\Theta_{i+1}}$. Examples of solvable structures will be found in [11]. They account for equations which appear to acquire extra symmetries under reduction. Examples have been studied in recent literature [1, 2] following an original example from Olver [13]. They have also been discussed in [11] where it is shown that the phenomenon of 'hidden symmetry' is due to such an extended notion of solvability.

4.5. *Some nonlinear ODEs*

The value of the above approach to the symmetry theory of ODEs is that it allows us, in principle, to display clearly the relationship between the existence of integrals within intermediate field extensions and non-characteristic symmetries within such extensions.

It is appropriate to start with the first-order, inhomogeneous equation

$$\tfrac{1}{2}z' + az + b = 0 \qquad (7)$$

where a and b belong to a differential field k. The general solution requires two Liouville extensions [12]. Thus we can extend the ideal generated by $\theta = \mathrm{d}z + 2(az + b)\,\mathrm{d}x$, $k(z)\langle \mathrm{d}z + 2(az+b)\,\mathrm{d}x\rangle \triangleleft k(z)\langle \mathrm{d}x, \mathrm{d}z\rangle$, to $k\langle\eta_1\rangle(z)\langle \mathrm{d}(z/\eta_1) + 2b/\eta_1\rangle \triangleleft k\langle\eta_1\rangle(z)\langle \mathrm{d}x, \mathrm{d}z\rangle$ by a Liouville extension using any solution η_1 to the homogeneous equation $\eta' + 2a\eta = 0$. The one-form generating the latter ideal is closed but only exact if b/η^2 has an integral in $k\langle\eta_1\rangle$ so, in general, we further extend to the generalized Liouville extension $k\langle\eta_1, \zeta_1\rangle$ where ζ_1 is any nonconstant solution to the second-order equation $\zeta'' - (b'/b + 2a)\zeta' = 0$.

Alternatively, let z_1 and z_2 be any linearly independent solutions of (7). (Note that because the equation is inhomogeneous, any pair of distinct solutions is linearly independent over $\mathbb{C}$.) We may take, for instance, $z_1 = \zeta_1\eta_1 + \eta_1$ and $z_2 = \zeta_1\eta_1 - \eta_1$. Then the general solution is $z = \lambda z_1 + (1-\lambda)z_2$ and an invariant can be taken to be $I = (z_1 - z_2)^{-1}(z - z_2)$. Any other choice for z_1 and z_2 induces an affine transformation of I. Consequently $v = (z_1 - z_2)\partial_z = 2\eta_1\partial_z$ is a non-characteristic symmetry and the form $\theta/(v\lfloor\theta)$ is the closed form above.

The Bernoulli equation

$$y' + ay + b/y = 0 \tag{8}$$

reduces to (7) under the quadratic map $z = y^2$ and so an invariant is

$$I = \frac{(y - y_2)(y + y_2)}{(y_1 - y_2)(y_1 + y_2)}.$$

An associated derivation is $v = (y_1^2 - y_2^2)/y\partial_y$ or, up to a constant, $v = \eta_1/y\partial_y$ for the reasons given above. So the Bernoulli equation has a symmetry with coefficients in a Liouville extension of k and integration proceeds as above.

For a more general Bernoulli equation

$$y' + ay + by^{n+1} = 0 \tag{9}$$

one uses the standard reduction to inhomogeneous linear form to obtain the invariant

$$I = \frac{y_1^n(y_2^n - y^n)}{y^n(y_2^n - y_1^n)}$$

and the corresponding derivations generated by

$$v = -\frac{y_2^n - y_1^n}{ny_1^n y_2^n} y^{n+1}\partial_y.$$

Here y_1 and y_2 are solutions of (9). v generates a Lie point symmetry of (9). In the case that a is constant (9) is a reduction of the second-order equation,

$$y'' = y^{-1}y'^2 + ng(x)y^n y' + g'(x)y^{n+1}$$

by the symmetry $y^{-1}\partial_y + (y^{-2}p + ng(x)y^{n-1})\partial_p$ which is not a Lie point symmetry. In [8] it is shown that there are no point symmetries for general $g(x)$.

In the case that b is constant solutions of (8) are also solutions of the well known Pinney equation,

$$y'' + \alpha y + \beta/y^3 = 0 \tag{10}$$

where $\alpha = a' - a^2$ and $\beta = b^2$. The ideal representing (8) can be seen to be the image of that representing (10):

$$\langle \mathrm{d}y + (ay + b/y)\,\mathrm{d}x\rangle = \Phi^*\langle \mathrm{d}y - p\,\mathrm{d}x, \mathrm{d}p + (\alpha y + \beta/y^3)\,\mathrm{d}x\rangle$$

under the restriction, Φ, to the submanifold $p + ay + b/y = 0$ of $\mathbb{C}^3$.

If y_1 and y_2 are solutions of (10) then the following are easily shown to be invariants belonging to the field $k_0\langle y_1, y_2\rangle(y, p)$:

$$I_2 = (yy_1' - py_1)^2 - \beta y_1^2/y^2 - \beta y^2/y_1^2$$
$$I_1 = (yy_2' - py_2)^2 - \beta y_2^2/y^2 - \beta y^2/y_2^2.$$

Elimination of p gives the general solution to (10) in an algebraic extension of the field $k_0 < y_1, y_2 >$. On the other hand we can also find invariants in the field $k_o < \phi_1, \phi_2 > (y, p)$ where ϕ_1 and ϕ_2 are solutions to the linear equation

$$\phi'' + \alpha\phi = 0$$

satisfying the Wronskian relation $\phi_1'\phi_2 - \phi_1\phi_2' = 1$, namely,

$$J_{ij} = (y\phi_i' - p\phi_i)(y\phi_j' - p\phi_j) - \beta\phi_i\phi_j/y^2$$

for $i, j = 1, 2$. These invariants are not algebraically independent. In fact $J_{11}J_{22} - J_{12}J_{21} = -\beta$. The general solution in a quadratic algebraic extension of $k_0\langle\phi_1, \phi_2\rangle$ gives the familiar expression [14] for the solution. Consequently $k_0\langle\phi_1, \phi_2\rangle$ is at most a subfield of $k_0\langle y_1, y_2\rangle$.

As a result the Pinney equation has non-characteristic symmetries,

$$v_i = 2(p\phi_i - y\phi_i')(\phi_i\partial_y + \phi_i'\partial_p) - 2\beta\phi_i^2/y^3\partial_p$$

for $i = 1, 2$ with coefficients in $k\langle\phi_1, \phi_2\rangle(y, p)$ and

$$\tilde{v}_i = 2(py_i - yy_i')(y_i\partial_y + y_i'\partial_p) - 2\beta(y_i^2/y^3 - y/y_i^2)\partial_p$$

for $i = 1, 2$ with coefficients in $k\langle y_1, y_2\rangle(y, p)$.

The classical Ermakov systems [15] have the form,

$$\begin{aligned} u'' + \alpha u + u^{-2}v^{-1}f(v/u) &= 0 \\ v'' + \alpha v + v^{-2}u^{-1}g(u/v) &= 0. \end{aligned} \tag{11}$$

These systems admit exact but implicit linearization [5] which, as in the case of the Pinney equation, makes use of the field extension $k_0\langle\phi_1, \phi_2\rangle$ where $\phi_1'' + \alpha\phi_1 = 0$. Invariants of the kind we seek were first constructed in an unpublished paper of Gordon [10] and subsequently by using the linearization [4]. Using $\mathrm{d}u - p\,\mathrm{d}x$ and $\mathrm{d}v - q\,\mathrm{d}x$ to describe the contact forms, they are of the form,

$$J_{ij} = (\phi_i q - \phi_i v)F_j(\rho) + v^{-1}(vp - uq)\phi_i\frac{\partial F_j}{\partial\rho}$$

where $\rho = u/v$ and F is a solution to a second-order linear equation depending on a parameter, J, the Lewis–Ray–Reid invariant. J is an invariant which is independent of ϕ_1 and ϕ_2. In fact the J_{ij} satisfy a quadratic relation analogous to that for the Pinney equation, namely,

$$J_{11}J_{22} - J_{12}J_{21} = J.$$

Consequently, derivations of these invariants will have coefficients depending on the ϕ_i's and will describe the symmetry algebra. They also have a ρ dependence determined by solutions of a second-order linear equation and hence a Liouville extension in functions of u and v. Some cases are effectively discussed in [4]. It is not clear whether there are invariants like the I_i for the Pinney equation expressed in term of solution pairs (u_i, v_i).

5. Conclusions

We have given in this paper a general framework for the discussion of symmetries of quite general ODEs, at least those defined by analytic forms, and we have illustrated how the description in terms of derivations over a field of invariants yields symmetries which lie in field extensions of the base field. This is not a constructive approach but it takes some of the mystery out of constructive approaches by replacing the question, ‘can we find a symmetry?’, with the question, ‘how big an extension must we use in order to construct a symmetry?’. It remains to push the analogy with Galois theory further, that is, to obtain a correspondence between subfields of the field of invariants and ideals of the full symmetry algebra.

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